TRINITY COLLEGE

Year 12 Mathematics Specialist
Test 2 2019

Section 1 Calculator Free
2D and 3D Vectors

STUDENT'S NAME Solutions

DATE: Friday 5 April **TIME:** 25 minutes **MARKS:** 25

INSTRUCTIONS:
Standard Items: Pens, pencils, drawing templates, eraser, formula page

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (3 marks)

Given $\underset{\sim}{a} = \begin{pmatrix} -8 \\ 4 \\ 16 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$, determine a vector in the direction of $\underset{\sim}{a}$ with magnitude $|\underset{\sim}{b}|$

note: $\underset{\sim}{a} = \begin{pmatrix} -8 \\ 4 \\ 16 \end{pmatrix}$ is parallel to $\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$

$\left|\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}\right| = \sqrt{4+1+16}$

$\left|\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}\right| = \sqrt{9+9+9}$

$$\underset{\wedge}{v} = \frac{\sqrt{27}}{\sqrt{21}} \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$$

$$= \frac{3\sqrt{3}}{\sqrt{21}} \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$$

✓ direction

✓ $|\underset{\sim}{a}|$ or $\sqrt{21}$

✓ $|\underset{\sim}{b}|$

2. (7 marks)

A plane contains the point $(1,2,3)$ and the vectors $\underset{\sim}{a} = \begin{pmatrix} -1 \\ -8 \\ 3 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix}$

(a) Determine a normal to the plane. [2]

$$\underset{\sim}{n} = \begin{pmatrix} -1 \\ -8 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} -16 - -15 \\ 3 - -2 \\ 5 - -8 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 5 \\ 13 \end{pmatrix}$$

$$\begin{array}{ccccc} i & j & k & i & j \\ -1 & -8 & 3 & -1 & -8 \\ 1 & -5 & 2 & 1 & -5 \end{array}$$

✓✓ −1 per error

(b) Determine the vector equation of the plane in scalar form. [2]

$$\underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{\sim}{a} \cdot \underset{\sim}{n}$$

$$\underset{\sim}{r} \cdot \begin{pmatrix} -1 \\ 5 \\ 13 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 5 \\ 13 \end{pmatrix}$$

✓ form

$$= -1 + 10 + 39$$

$$\underset{\sim}{r} \cdot \begin{pmatrix} -1 \\ 5 \\ 13 \end{pmatrix} = 48$$

✓ final eqn

(c) The line $\underset{\sim}{r} = \begin{pmatrix} 10 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ intersects the plane. Determine the point of intersection. [3]

$\Rightarrow$ line also satisfies eqn of plane

$$\Rightarrow \begin{pmatrix} 10+2\lambda \\ 3+\lambda \\ 1-\lambda \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 5 \\ 13 \end{pmatrix} = 48$$

✓ substitution

$$\Rightarrow -10 - 2\lambda + 15 + 5\lambda + 13 - 13\lambda = 48$$

$$\Rightarrow -10\lambda + 18 = 48$$

$$\Rightarrow -10\lambda = 30$$

$$\Rightarrow \lambda = -3$$

✓ λ value

$$\therefore \text{pt is } \underset{\sim}{r} = \begin{pmatrix} 10 \\ 3 \\ 1 \end{pmatrix} - 3 \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

✓ point

$$= \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix}$$

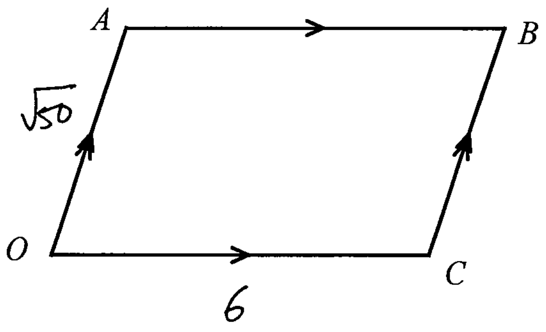
3. (6 marks)

A parallelogram has the following information:

$\overrightarrow{OA} = 5\underset{\sim}{i} + 5\underset{\sim}{j}$

$\overrightarrow{OC} = 6\underset{\sim}{i}$

Where O is the origin.



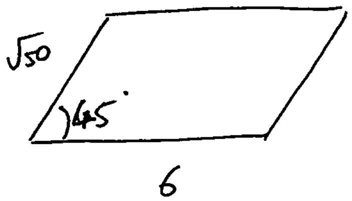
(a) Determine the position vector of B [2]

$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$ ✓ eqn

$= \begin{pmatrix} 5 \\ 5 \end{pmatrix} + \begin{pmatrix} 6 \\ 0 \end{pmatrix}$ as $\overrightarrow{AB} \parallel \overrightarrow{OC}$

$= \begin{pmatrix} 11 \\ 5 \end{pmatrix}$ ✓ vector

(b) Determine the area of the parallelogram [4]



$|\overrightarrow{OA}| = \sqrt{50}$

makes an angle of 45° to x-axis ✓

$\therefore A = 2 \times \Delta s$ ✓ ✓

$= 2 \times \frac{1}{2} \times 6 \times \sqrt{50} \times \sin 45^\circ$

$= 6\sqrt{50}\,\frac{\sqrt{2}}{2}$

$= 3\sqrt{100}$ ✓

$= 30$ units^2

OR

Use cross product

→ introduce 30

$\overrightarrow{OA} = \begin{pmatrix} 5 \\ 5 \\ 0 \end{pmatrix}$

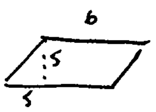
$\overrightarrow{OC} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$

$\therefore$ Area $= \left| \begin{pmatrix} 5 \\ 5 \\ 0 \end{pmatrix} \times \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} \right|$

$= \left| \begin{pmatrix} 0 \\ 0 \\ -30 \end{pmatrix} \right|$ //

$= 30$ units^2 //

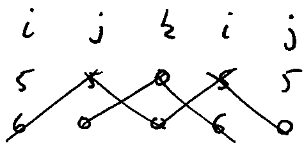
OR



$A = b \times h$

$= 6 \times 5$

$= 30$ units^2



4. (9 marks)

(a) Consider A, B and C with position vectors $\begin{array}{ccc} A & B & C \\ \begin{pmatrix}4\\-3\\5\end{pmatrix}, & \begin{pmatrix}6\\7\\9\end{pmatrix} \text{ and} & \begin{pmatrix}7\\2\\6\end{pmatrix}\end{array}$ respectively. Show that these points form the vertices of an isosceles triangle.

[6]

Isosceles → two equal lengths

$|\vec{AB}| = \left|\begin{pmatrix}2\\10\\4\end{pmatrix}\right|$ ✓ $\qquad |\vec{AC}| = \left|\begin{pmatrix}3\\5\\+1\end{pmatrix}\right|$ ✓

$= \sqrt{120}$ ✓ $\qquad = \sqrt{35}$ ✓

$|\vec{BC}| = \left|\begin{pmatrix}1\\-5\\-3\end{pmatrix}\right|$ ✓

$= \sqrt{35}$ ✓

As $|\vec{AC}| = |\vec{BC}|$ it is an isosceles triangle

(b) Determine the Cartesian equation of the plane that contains the isosceles triangle. [3]

use scalar product form

i j k i j
3 ~~5~~ ~~1~~ 3 5
1 -5 3 1 -5

$n = \begin{pmatrix}3\\5\\-1\end{pmatrix} \times \begin{pmatrix}1\\-5\\3\end{pmatrix} = \begin{pmatrix}15-5\\-1-9\\-15-5\end{pmatrix} = \begin{pmatrix}10\\-10\\-20\end{pmatrix}$ ✓

This is // to $\begin{pmatrix}1\\-1\\-2\end{pmatrix}$

$\underset{\sim}{r} \cdot \hat{n} = \underset{\sim}{a} \cdot \hat{n}$

$\underset{\sim}{r} \cdot \begin{pmatrix}1\\-1\\2\end{pmatrix} = \begin{pmatrix}4\\-3\\5\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\2\end{pmatrix}$ ✓

$= 17$

✓

$\therefore \; x - y + 2z = 17$

TRINITY COLLEGE

Year 12 Mathematics Specialist
Test 2 2019

Section 2 Calculator Assumed
2D and 3D Vectors

STUDENT'S NAME ______________________________

DATE: Friday 5 April **TIME:** 25 minutes **MARKS:** 25

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser, formula page

Special Items: Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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5. (5 marks)

Two particles A and B have initial position vectors of $-73\underset{\sim}{i}+242\underset{\sim}{j}-476\underset{\sim}{k}$ and $199\underset{\sim}{i}-30\underset{\sim}{j}+170\underset{\sim}{k}$ respectively and their velocity vectors are $4\underset{\sim}{i}-13\underset{\sim}{j}+29\underset{\sim}{k}$ and $-12\underset{\sim}{i}+3\underset{\sim}{j}-9\underset{\sim}{k}$ respectively. All units are S.I. units (meters and seconds).

(a) Determine the position vector where the two paths intersect. [3]

$$\begin{pmatrix} -73 \\ 242 \\ -476 \end{pmatrix} + t_1 \begin{pmatrix} 4 \\ -13 \\ 29 \end{pmatrix} = \begin{pmatrix} 199 \\ -30 \\ 170 \end{pmatrix} + t_2 \begin{pmatrix} -12 \\ 3 \\ -9 \end{pmatrix} \checkmark$$

$$\Rightarrow \quad t_1 = 17, \quad t_2 = 17 \checkmark \quad \therefore \text{ intersect at } \begin{pmatrix} -5 \\ 21 \\ 17 \end{pmatrix} \checkmark$$

(b) Do the two particles collide? Explain. [2]

Yes, at the same ✓ spot at the same time ✓

6. (8 marks)

A and B have position vectors of $\overset{A}{\begin{pmatrix} 2 \\ -6 \end{pmatrix}}$ and $\overset{B}{\begin{pmatrix} -3 \\ -5 \end{pmatrix}}$ respectively. A line passes through the two points.

(a) Determine the vector equation of the line. [2]

$$\overrightarrow{AB} = \begin{pmatrix} -5 \\ 1 \end{pmatrix} \quad \checkmark$$

$$\therefore \quad \underset{\sim}{r} = \begin{pmatrix} 2 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 1 \end{pmatrix} \quad \checkmark$$

(b) Determine the parametric equations of the line. [2]

$$x = 2 - 5\lambda \quad \checkmark$$

$$y = -6 + \lambda \quad \checkmark$$

(c) Determine the Cartesian equation of the line. [2]

$$\frac{x-2}{-5} = y + 6 \quad \checkmark\checkmark$$

(d) Determine the angle the line makes with the y-axis. [2]

line has direction $\begin{pmatrix} -5 \\ 1 \end{pmatrix}$

y-axis has direction $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\text{angle}\left(\begin{pmatrix} -5 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = 78.7^\circ$$

$\checkmark$ $\checkmark$

7. (7 marks)

A sphere has the vector equation $\left| \underset{\sim}{r} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right| = 5$

(a) Determine where the line $\underset{\sim}{r} = \begin{pmatrix} 1 \\ -26 \\ 24 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 4 \\ -3 \end{pmatrix}$ intersects the sphere. [4]

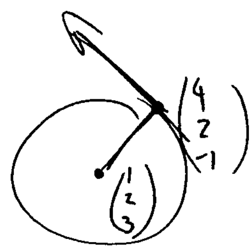
$\Rightarrow \left| \begin{pmatrix} 1 \\ -26+4\lambda \\ 24-3\lambda \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right| = 5$ ✓

Solving $\lambda = 6, 8$ ✓

$\therefore$ pts $\begin{pmatrix} 1 \\ -2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 6 \\ 0 \end{pmatrix}$

✓ ✓

(b) The point $\begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$ lies on the surface of the sphere. Determine the equation of a line that is tangent to the sphere at this point. [3]



Need a $\perp$ direction to radius

rad $= \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}$ ✓

$\therefore \perp$ dir $= \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$ ✓

$\therefore$ line $\underset{\sim}{r} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$ ✓

8. (5 marks)

The points $A(1,-1,3)$, $B(4,1,-2)$, $C(-1,-1,1)$ and $D(1,1,1)$ all lie on the surface of a sphere.

(a) Determine the centre and radius of the sphere. [3]

All points must satisfy Cartesian eqn $(x+a)^2+(y+b)^2+(z+c)^2=d$

$$\Rightarrow \quad \begin{aligned} (1+a)^2+(-1+b)^2+(3+c)^2 &= d \\ (4+a)^2+(1+b)^2+(-2+c)^2 &= d \\ (-1+a)^2+(-1+b)^2+(1+c)^2 &= d \\ (1+a)^2+(1+b)^2+(1+c)^2 &= d \end{aligned} \quad \checkmark$$

Solving we get

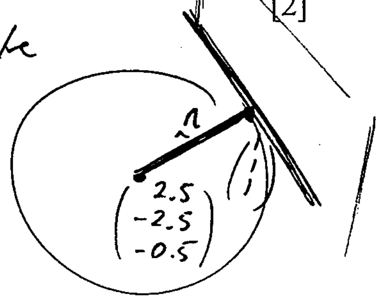
$$(x-2.5)^2+(y+2.5)^2+(z+0.5)^2 = 16.75$$

$\therefore$ Centre $(2.5, -2.5, -0.5)$ ✓

radius $\sqrt{16.75} \approx 4.09$ ✓

(b) Determine the equation of the plane that is tangent to the sphere at the point $D(1,1,1)$ [2]

normal to plane will go through centre



$$\Rightarrow \quad \underset{\sim}{n} = \begin{pmatrix} -1.5 \\ +3.5 \\ 1.5 \end{pmatrix} \quad \checkmark$$

$$\therefore \text{ Plane } \quad \underset{\sim}{r} \cdot \begin{pmatrix} -1.5 \\ 3.5 \\ 1.5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1.5 \\ 3.5 \\ 1.5 \end{pmatrix}$$

$$\underset{\sim}{r} \cdot \begin{pmatrix} -1.5 \\ 3.5 \\ 1.5 \end{pmatrix} = 3.5 \quad \checkmark$$